SOLUTION OF A GENERALIZED RIEMANN BOUNDARY-VALUE PROBLEM FOR TWO CIRCULAR INCLUSIONS

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Introduction. Consider a three-phase plane system composed of two simply-connected domains S_1, S_2 confined by two non-intersecting circumferences $\gamma_1 = \{z : |z| = r_1\}$, $\gamma_2 = \{z : |z - l| = r_2\}$ and a double-connected domain $S_3 = \Re^2 \setminus \{\bar{S}_1 \cup \bar{S}_2\}$. Designate the hydraulic heads as $h(x, y)$, the specific discharge vectors as $\vec{v}(x, y) = (v_x, v_y)$ and the conductivity as $k(x, y) \equiv k_j$, $z \in S_j$, $j = 1,2,3$. According to Darcy's law $\vec{v}(x, y) = \nabla h(x, y)$. Introduce the complex coordinate $z = x + iy$, and complex velocities $\vec{v}(x, y) = v_x + iv_y$. Thus, the complex function $v(z) = v_x - iv_y = dw/dz$ ($w(z)$ is a complex potential) is holomorphic within each of three isotropic phases S_j. The refraction conditions

$$\vec{v}_{3\tau} / k_3 = \vec{v}_{1,2\tau} / k_{1,2} \tag{1}$$

hold in terms of normal ($\vec{v}_\nu$) and tangential ($\vec{v}_\tau$) velocity components along all the boundaries separating different phases:

Different special cases of the problem (1) were studied ([1]-[9]). All these cases follow from our solution below.

Two-lens System. At first, we consider a case when circles S_1, S_2 are placed outside (Fig. 1a) each other. A limiting situation when one of the circles is a half-plane is shown in Fig. 1b.

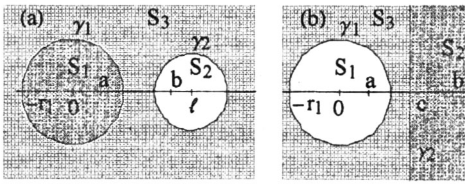

Figure 1.

In terms of the piece-wise holomorphic function $v(z) = v_j(z) \in \mathcal{H}(S_j)$ the problem (1) is equivalent to

$$v_3(t) = A_j v_j(t) - B_j [t'(s)]^{-2} \overline{v_j(t)}, \quad t \in \gamma_j, \quad j = 1,2, \tag{2}$$

where $t'(s)$ is the derivative of the function of a pint of the contour γ_j with respect to its natural parameter s,

$$A_j = \frac{k_3 + k_j}{2k_j}, \quad B_j = \frac{k_j - k_3}{2k_j}. \tag{3}$$

At the infinity the function $v_3(z)$ is bounded and

$$v_3(\infty) = V_0 = V_x - iV_y. \tag{4}$$

Outline the solution of the problem (2) - (4). Let the points a,b be symmetric about both circles γ_1, γ_2 (Fig. 1a). The function $\varsigma = (z-a)/(z-b)$ maps conformally the asymmetric ring s_3 onto a concentric one $S_3^* = \{\zeta: R_1 < |\zeta| < R_2\}$. The domains S_1, S_2 are mapped onto $S_1^* = \{\zeta : |\zeta| < R_1\}$, and $S_2^* = \{\zeta : |\zeta| > R_2\}$, where $R_1^2 = a/b$ and $R_2^2 = (l-a)/(l-b)$ $(0 < R_1 < 1 < R_2)$. For a piece-wise holomorphic function $V(\varsigma) = V_j(\varsigma) = v_j(T(\zeta)), \zeta \in S_j^*$ $j = 1,2,3$, from (2) the following boundary-value problem can be stated

$$V_3(t) = A_j V_j(t) + B_j \left(\frac{R_j(t-1)}{t - R_j^2} \right)^2 \overline{V_j(t)}. \tag{5}$$

The conditions (5) hold for $t \in \Gamma_j = \{t : |t| = R_j\}$, $l = 1,2$. From (4) we have $V_3(1) = V_0$ and $V_2(\infty) = v_2(b) \neq \infty$. From the Laurant theorem $V_3(\zeta) = V_3^1(\zeta) + V_3^2(\zeta)$, where the functions $V_3^1(\zeta)$ $(V_3^1(1) = V_0)$ and $V_3^2(\zeta)$ $(V_3^2(1) = 0)$ are holomorphic in the circle of radius R_2 and outside the circle of radius R_1, correspondingly. Consider the following pair of functions

$$\Phi_j(\zeta) = V_3^j(\zeta) - A_j V_j(\zeta) \ \text{ if } |\zeta|^{3-2j} \leq R_j^{3-2j},$$

$$\Phi_j(\zeta) = B_j R_j^2 \overline{\left(\frac{\zeta - 1}{\zeta - R_j^2} \right)^2 V_j\left(\frac{R_j^2}{\overline{\zeta}} \right)} - V_3^{3-j} \ \text{ if } |\zeta|^{3-2j} \geq R_j^{3-2j}.$$

After some algebra we obtain $\Phi_1(\zeta) \equiv V_3^2(1) = 0$ and $\Phi_2(\zeta) \equiv -V_3^1(1) = -V_0$. Excluding from the obtained identities V_1, V_2 and V_3^2 we get the following functional relationship with respect to V_3^1:

$$V_3^1(\zeta) = V_0 + \overline{V}_0 \Delta_2 R_2^2 \left(\frac{\zeta - 1}{\zeta - R_2^2} \right)^2 + g\delta \left(\frac{1-\zeta}{1 - g\zeta} \right)^2 V_3^1(g\zeta),$$

where $g = R_1^2/R_2^2$, $\Delta_{1,2} = B_{1,2}/A_{1,2}$, $\delta = \Delta_1 \Delta_2$. Solving the last equation we determine $V_3^1(\zeta)$, and consequently, all other unknown functions. Thus, we prove

Theorem 1. *The problem* (2) - (4) *is unconditionally and uniquely solvable if* $k_j \neq o, \infty$ *and* $k_j \neq k_n$ *for* $j \neq n$, $j,n = \overline{1,3}$. *The solution is determined by the formulae*

$$\begin{gathered} v_j(z) = \left(V_0 + \sigma_j(z) \right) / A_j, \quad j = 1,2, \\ v_3(z) = V_0 + \sigma_1(z) + \sigma_2(z), \end{gathered} \tag{6}$$

$$\sigma_1(z)=\sum_{j=1}^{\infty}\delta^j\frac{V_0\Lambda_j}{(z-b_{1j})^2}+\Delta_2\sum_{j=0}^{\infty}\delta^j\frac{\overline{V}_0\Lambda_{1j}}{(z-b_{2j})^2}, \tag{7}$$

$$\sigma_2(z)=\sum_{j=1}^{\infty}\delta^j\frac{V_0\Lambda_j}{(z-a_{1j})^2}+\Delta_1\sum_{j=0}^{\infty}\delta^j\frac{\overline{V}_0\Lambda_{2j}}{(z-a_{2j})^2}, \tag{8}$$

where $g=[a(l-b)]/[b(l-a)]$, $T(\zeta)=(b\zeta-a)/(\zeta-1)$,

$$a_{1j}=T(g^j),\qquad a_{2j}=T(R_1^2g^j),$$
$$b_{1j}=T(g^{-j}),\qquad b_{2j}=T(R_2^2g^{-j}), \tag{9}$$

$$\Lambda_j=\chi(g^j),\qquad \Lambda_{1j}=\chi(R_2^2g^{-j}),$$
$$\Lambda_{2j}=\chi(R_1^2g^j),\quad \chi(x)=(b-a)^2x/(1-x)^2. \tag{10}$$

Remark 1. Solution of the problem (2) - (4) with real coefficients (3) can be readily generalised to the case of complex coefficients k_j occurring in electrodynamics.

Remark 2. The known solution for two impermeable cylinders can be obtained from Theorem 1 as a limiting case for $k_3\to\infty$ or $k_1=k_2\to 0$. Here $v_1(z)\equiv v_2(z)\equiv 0$ and $v_3(z)$ follows from (6) - (10), where $\Delta_1=\Delta_2=-\delta=1$.

Solution of the problem (2) - (4) for the case shown in Fig. 1b follows from (6) - (10) at $r_2\to\infty$ and at a fixed distance between the inclusions S_1 and S_2. In particular, the following theorem is proved

Theorem 2. *If* S_1 *is a circle of radius* r_1 *centred at the origin of coordinates and* $S_2=\{z:\operatorname{Re}z>c>r_1\}$, $S_3=\{z:|\operatorname{Re}z|<c,\ |z|>r_1\}$, *then a solution to* (2) - (4) *is*

$$v_1(z)=(V_0+\sigma_1(z))/A_1,$$
$$v_2(z)=V_xk_2/k_3-iV_y+\sigma_2(z)/A_2, \tag{11}$$
$$v_3(z)=V_0+\sigma_1(z)+\sigma_2(z),$$

$$\sigma_1(z)=V_0\sum_{j=1}^{\infty}\frac{\delta^j\Lambda_j}{(z-b_j)^2},\quad \sigma_2(z)=\frac{\overline{V}_0}{\Delta_2}\sum_{j=1}^{\infty}\frac{\delta^j\Lambda_j}{(z-a_j)^2}, \tag{12}$$

$$a_j=T(r^{2j}),\ b_j=T(r^{-2j}),\ r=(c-\sqrt{c^2-r_1^2})/r_1,$$
$$\Lambda_j=4(c^2-r_1^2)^2r^{2j}/(1-r^{2j})^2.$$

At $l-r_1-r_2\to 0$ the two circular inclusions S_1,S_2 touch each other and the following theorem is valid.

Theorem 3. *Problem* (2) - (4) *for the case of two circular tangent inclusions* ($l=r_1+r_2$) *at* $k_j\neq 0,\infty$ *and* $k_j\neq k_n$ *for* $j\neq n$, $j,n=\overline{1,3}$, *has a unique solution defined by the formulae* (6) - (8), *where*

$$\Lambda_j = \left(\frac{r_1}{j(1+\theta)}\right)^2, \Lambda_{nj} = \left(\frac{r_1}{(j+2-n)\theta + j+n-1}\right)^2,$$

$$a_{1j} = T(1-g_j),\quad a_{2j} = T(-1-g_j),\ b_{1j} = T(1+g_j),\ b_{2j} = T(g_{j+1}),$$

and $T(\zeta) = r_1(\zeta+1)/(\zeta-1)$, $g_j = 2j(1+\theta)$, $\theta = r_1/r_2$.

Theorem 4. *The unique solution of the problem* (2) - (4) *for the case when* $S_1 = \{z : |z| < r\}$, $S_2 = \{z : \operatorname{Re} z > r\}$, *and* $S_3 = \{z : |z| > r, \operatorname{Re} z < r\}$ *for all non-limiting and non-degenerating cases is determined by the relations* (11), (12) *where*

$$\Lambda_j = r/j,\ a_j = r - r/j,\ b_j = r + r/j.$$

Remark 1. The case of two impermeable cylinders touching each other can be retrieved from Theorem 3 substituting z by $z+r$ and putting $k_1 = k_2 = 0$. Then

$$v_3(z) = \sum_{j=-\infty}^{\infty}\left(\frac{V_0\theta^2}{(\theta+jz)^2} - \frac{\overline{V}_0\theta^2}{(\theta+(j+\theta/r_1)z)^2}\right), \tag{13}$$

where $\theta = r_1 r_2/(r_1 + r_2)$.

It follows from (13) at $r_1 = r_2 = r$ that

$$v_3(z) = \left(\frac{\pi r}{2z}\right)^2\left(V_0 \csc^2\frac{\pi r}{2z} - \overline{V}_0 \sec^2\frac{\pi r}{2z}\right).$$

For absolutely permeable inclusions $k_1 = k_2 = \infty$ the corresponding solution in S_3 differs from the written above only in the sign at $\overline{V}_0$, because for equipotential circles $\Delta_1 = \Delta_2 = 1$. For example ([9]), when the imposed field is oriented orthogonally to the line passing through the inclusion centres ($V_0 = iV_y$), the corresponding complex potential is

$$w(z) = i\pi r V_y \csc\frac{\pi r}{z}.$$

Remark 2. A classical problem for a flow around an impermeable cylinder placed above an impermeable horizontal bottom ([7]) can be obtained from Theorem 4. Substituting for $z-r$ by iz, V_0 and $v(z)$ by iV_0 and $iv(z)$ respectively, at the limit $k_1, k_2 \to 0$

$$v_3(z) = V_0\left(\frac{z}{\pi r}\sinh\frac{\pi r}{z}\right)^{-2}.$$

Corresponding complex potential is

$$w_3(z) = V_0 \pi r \coth(\pi r/z).$$

Eccentric Annulus . We derive a solution to (2) - (4) for an annulus, which boundaries γ_1, γ_2 are either separated (Fig. 2a) or touch each other (Fig. 2b). Thus two connected domain S_3 is bounded here and an additional condition is imposed on V_2

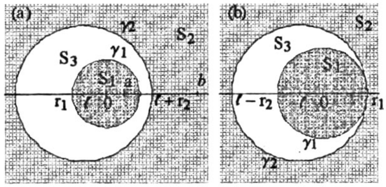

Figure 2.

$$V_2(\infty) = V_0. \tag{14}$$

We proved the following theorem:

Theorem 5. *The unique solution of problem* (2), (14) *for the structure in* Fig. 2a *can be written in non-limiting cases as*

$$v_1(z) = (1+\Delta_1)(1-\Delta_2)(V_0 + \sigma_1(z))_1,$$

$$v_2(z) = V_0 - \overline{V}_0 \frac{\Delta_2 r_2^2}{(z-l)^2} + (1+\Delta_2)\sigma_2(z),$$

$$v_3(z) = (1-\Delta_2)V_0 + \sigma_1(z) + \sigma_2(z),$$

where

$$\sigma_1(z) = V_0 \sum_{j=1}^{\infty} \frac{\delta^j \Lambda_{1j}}{(z-b_j)^2}, \qquad \sigma_2(z) = \overline{V}_0 \Delta_1 (1-\Delta_2) \sum_{j=0}^{\infty} \frac{\delta^j \Lambda_{2j}}{(z-a_j)^2},$$

$a_j = a_{2j}$, $b_j = b_{1j}$ *and the values of* $\Lambda_{1j}, \Lambda_{2j}$ *are derived according to relations* (9) *and* (10), *correspondingly.*

For the structure in Fig. 2b the following result is proved

Theorem 6. *Problem* (2), (14) at $k_j \neq 0, \infty$ *and* $k_j \neq k_n$ *for* $j \neq n$, $j, n = \overline{1,3}$, *in the case of touching circumferences* γ_1, γ_2 *has a unique solution*

$$v_1(z) = (1+\Delta_1)((1-\Delta_2)V_0 + \sigma_1(z)),$$

$$v_2(z) = V_0 + (1+\Delta_2)\left(\overline{V}_0 \frac{\Delta_2 r_2^2}{(z - r_1 + r_2)^2} + \sigma_2(z)\right), \tag{17}$$

$$v_3(z) = (1-\Delta_2)V_0 + \sigma_1(z) + \sigma_2(z),$$

$$\sigma_1(z) = V_0(1-\Delta_2) \sum_{j=1}^{\infty} \frac{\delta^j \Lambda_{1j}^2}{(z-b_j)^2}, \quad \sigma_2(z) = \overline{V}_0 \Delta_1 (1-\Delta_2) \sum_{j=1}^{\infty} \frac{\delta^j \Lambda_{2j}^2}{(z-a_j)^2},$$

$$\Lambda_{2j} = r_1 j / ((1-\theta)j + 1), \quad \Lambda_{1j} = r_1 / [((1-\theta)j], \quad \theta = r_1 / r_2,$$

$$a_j = T(-1-g_p), b_j = T(1+g_p), \; g_j = 2j(1-\theta), \; T(\zeta) = r_1(\zeta+1)/(\zeta-1).$$

In conclusion we note that truncated series $v_{jN}(z)$ are used for practical calculations i.e. the sum of first N-1 terms of the solutions obtained above.

Exact estimations for the remainders $|v_j(z)-v_{jN}(z)|$ are derived. For example, next estimations are valid for the solution (6) - (8):

$$|v_j(z)-v_{jN}(z)|\leq C_j\,|\delta|^N\, g^{N-1}, \qquad j=1,2,3.$$

and correspondingly

$$C_1\leq r_1^2\frac{|V_0|}{A_1}\frac{r_2^2/\ell^2+g|\Delta_2|}{(1-g|\delta|)(b-r_1)^2},$$

$$C_2\leq r_2^2\frac{|V_0|}{A_1}\frac{r_1^2/\ell^2+g|\Delta_1|}{(1-g|\delta|)(\ell-r_2-a)^2},$$

$$C_3\leq\frac{|V_0|}{(1-g|\delta|)}\left(r_1^2\frac{r_2^2/\ell^2+g|\Delta_2|}{(r_1-a)^2}+r_2^2\frac{r_1^2/\ell^2+g|\Delta_1|}{(b+r_2-\ell)^2}\right).$$

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